

THE AUTOMORPHISM GROUP OF A VARIETY WITH TORUS ACTION OF COMPLEXITY ONE

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ABSTRACT. We determine the root system of the automorphism group of a complete variety with a torus action of complexity one.

1. INTRODUCTION

Demazure [5] and Cox [4] investigated the automorphism group of a complete normal toric variety and gave a description of the root system in terms of the defining combinatorial data, i.e. the fan. The aim of this article is to extend these results to the more general case of a normal complete rational variety X coming with an effective torus action $T \times X \rightarrow X$ of complexity one, i.e. the dimension of T is one less than that of X .

Our approach to the automorphism group of X goes via the Cox ring $\mathcal{R}(X)$ which can be defined for any normal complete variety with finitely generated divisor class group $\text{Cl}(X)$. The presence of the complexity one torus action implies that $\mathcal{R}(X)$ is of a quite special nature: generators, relations as well as the $\text{Cl}(X)$ -grading can be encoded in a sequence $A = a_0, \dots, a_r$ of pairwise linearly independent vectors in \mathbb{K}^2 and an integral matrix

$$P = \begin{bmatrix} -l_0 & l_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \dots & l_r & 0 \\ d_0 & d_1 & \dots & d_r & d' \end{bmatrix}$$

of size $(n+m) \times (r+s)$, where l_i are nonnegative integral vectors of length n_i , the d_i are $s \times n_i$ blocks, d' is an $s \times m$ block and the columns of P are pairwise different primitive vectors generating the column space \mathbb{Q}^{r+s} as a convex cone. Conversely, the data A, P always define a Cox ring $\mathcal{R}(X) = R(A, P)$ of a complexity one T -variety X . The dimension of X equals $s+1$ and the acting torus T has \mathbb{Z}^s as its character lattice. The matrix P determines the grading and the exponents occurring in the relations, whereas A is responsible for continuous aspects. For details, we refer to Section 3 and the original literature [9, 10].

The crucial notion for the investigation of the automorphism group $\text{Aut}(X)$ are the *Demazure P -roots*, see Definition 5.1. Roughly speaking, these are finitely many integral linear forms u on \mathbb{Z}^{r+s} satisfying a couple of linear inequalities on the columns of P . In particular, given P , the Demazure P -roots can be easily determined. Our main result expresses the root system of the automorphism group $\text{Aut}(X)$ and, moreover, the approach shows how to obtain the corresponding root subgroups, see Theorem 5.4 and Corollary 5.10 for the precise formulation:

Theorem. *Let X be a nontoric normal complete rational variety with an effective torus action $T \times X \rightarrow X$ of complexity one. Then $\text{Aut}(X)$ is a linear algebraic group having T as a maximal torus and the roots of $\text{Aut}(X)$ with respect to T are precisely the \mathbb{Z}^s -parts of the Demazure P -roots.*

The basic idea of the proof is to relate the group $\text{Aut}(X)$ to the group of graded automorphisms of the Cox ring. This is done in Section 2 more generally for arbitrary Mori dream spaces, i.e. normal complete varieties with a finitely generated Cox ring $\mathcal{R}(X)$. In this setting, the grading by the divisor class group $\text{Cl}(X)$ defines an action of the characteristic quasitorus $H_X = \text{Spec } \mathbb{K}[\text{Cl}(X)]$ on the total coordinate space $\overline{X} = \text{Spec } \mathcal{R}(X)$ and X is the quotient of an open subset $\widehat{X} \subseteq \overline{X}$ by the action of H_X . The group of $\text{Cl}(X)$ -graded automorphisms of $\mathcal{R}(X)$ is isomorphic to the group $\text{Aut}(\overline{X}, H_X)$ of H_X -equivariant automorphisms. Moreover, the group $\text{Bir}_2(X)$ of birational automorphisms of X defined on an open subset of X having complement of codimension at least two plays a role. Theorem 2.1 brings all groups together:

Theorem. *Let X be a Mori dream space. Then there is a commutative diagram of morphisms of linear algebraic groups where the rows are exact sequences and the upwards inclusions are of finite index:*

$$\begin{array}{ccccccc}
 1 & \longrightarrow & H_X & \longrightarrow & \text{Aut}(\overline{X}, H_X) & \longrightarrow & \text{Bir}_2(X) \longrightarrow 1 \\
 & & \parallel & & \uparrow & & \uparrow \\
 1 & \longrightarrow & H_X & \longrightarrow & \text{Aut}(\widehat{X}, H_X) & \longrightarrow & \text{Aut}(X) \longrightarrow 1
 \end{array}$$

This means in particular that the unit component of $\text{Aut}(X)$ coincides with that of $\text{Bir}_2(X)$ which in turn is determined by $\text{Aut}(\overline{X}, H_X)$, the group of graded automorphisms of the Cox ring. Coming back to rational varieties X with torus action of complexity one, the task then is a detailed study of the graded automorphism group of the rings $\mathcal{R}(X) = R(A, P)$. This is done in a purely algebraic way. The basic concepts are provided in Section 3. The key result is the description of the “primitive homogeneous locally nilpotent derivations” on $R(A, P)$ given in Theorem 4.3. The proof of the main theorem in Section 5 then relates the Demazure P -roots via these derivations to the roots of the automorphism group $\text{Aut}(X)$.

In Section 6 we apply our results to the study of almost homogeneous (possibly singular) Mori dream surfaces X , i.e. complete normal surfaces with a finitely generated Cox ring such that $\text{Aut}(X)$ has an open orbit in X . Such a surface X is either toric, or $\text{Aut}(X)^0$ is unipotent, or the maximal torus of $\text{Aut}(X)$ is of dimension one, see Proposition 6.1. We consider the latter setting and take a closer look to the case of Picard number one. It turns out that these surfaces are always (possibly singular) del Pezzo surfaces and, up to isomorphism, there are countably many of them, see Corollary 6.4. Finally in the case that X is log terminal with only one singularity, we give classifications for fixed Gorenstein index.

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2. THE AUTOMORPHISM GROUP OF A MORI DREAM SPACE

Let X be a normal complete variety defined over an algebraically closed field \mathbb{K} of characteristic zero with finitely generated divisor class group $\text{Cl}(X)$ and Cox

sheaf \mathcal{R} ; we recall the definition below. If X is a Mori dream space, i.e. the Cox ring $\mathcal{R}(X) = \Gamma(X, \mathcal{R})$ is finitely generated as a \mathbb{K} -algebra, then we obtain the following picture

$$\begin{array}{ccccc} \mathrm{Spec}_X \mathcal{R} & = & \widehat{X} & \subseteq & \overline{X} & = & \mathrm{Spec} \mathcal{R}(X) \\ & & \downarrow \scriptstyle // H_X & & & & \\ & & X & & & & \end{array}$$

where the *total coordinate space* \overline{X} comes with an action of the *characteristic quasitorus* $H_X := \mathrm{Spec} \mathbb{K}[\mathrm{Cl}(X)]$, the *characteristic space* \widehat{X} , i.e. the relative spectrum of the Cox sheaf, occurs as an open H_X -invariant subset of \overline{X} and the map $p_X: \widehat{X} \rightarrow X$ is a good quotient for the action of H_X .

We study automorphisms of X in terms of automorphisms of \overline{X} and \widehat{X} . By an H_X -equivariant automorphism of \overline{X} we mean a pair $(\varphi, \tilde{\varphi})$, where $\varphi: \overline{X} \rightarrow \overline{X}$ is an automorphism of varieties and $\tilde{\varphi}: H_X \rightarrow H_X$ is an automorphism of linear algebraic groups satisfying

$$\varphi(t \cdot x) = \tilde{\varphi}(t) \cdot \varphi(x) \quad \text{for all } x \in \overline{X}, t \in H_X.$$

We denote the group of H_X -equivariant automorphisms of \overline{X} by $\mathrm{Aut}(\overline{X}, H_X)$. Analogously, one defines the group $\mathrm{Aut}(\widehat{X}, H_X)$ of H_X -equivariant automorphisms of \widehat{X} . A *weak automorphism* of X is a birational map $\varphi: X \rightarrow X$ which defines an isomorphism of big open subsets, i.e., there are open subsets $U_1, U_2 \subseteq X$ with complement $X \setminus U_i$ of codimension at least two in X such that $\varphi|_{U_1}: U_1 \rightarrow U_2$ is a regular isomorphism. We denote the group of weak automorphisms of X by $\mathrm{Bir}_2(X)$.

Theorem 2.1. *Let X be a Mori dream space. Then there is a commutative diagram of morphisms of linear algebraic groups where the rows are exact sequences and the upwards inclusions are of finite index:*

$$\begin{array}{ccccccc} 1 & \longrightarrow & H_X & \longrightarrow & \mathrm{Aut}(\overline{X}, H_X) & \longrightarrow & \mathrm{Bir}_2(X) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & H_X & \longrightarrow & \mathrm{Aut}(\widehat{X}, H_X) & \longrightarrow & \mathrm{Aut}(X) \longrightarrow 1 \end{array}$$

Moreover, there is a big open subset $U \subseteq X$ with $\mathrm{Aut}(U) = \mathrm{Bir}_2(X)$ and the groups $\mathrm{Aut}(\overline{X}, H_X)$, $\mathrm{Bir}_2(X)$, $\mathrm{Aut}(\widehat{X}, H_X)$, $\mathrm{Aut}(X)$ act morphically on \overline{X} , U , \widehat{X} , X , respectively.

Our proof uses some ingredients from algebra which we develop first. Let K be a finitely generated abelian group and consider a finitely generated integral \mathbb{K} -algebra

$$R = \bigoplus_{w \in K} R_w.$$

The *weight monoid* of R is the submonoid $S \subseteq K$ consisting of the elements $w \in K$ with $R_w \neq 0$. The *weight cone* of R is the convex cone $\omega \subseteq K_{\mathbb{Q}}$ in the rational vector space $K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the weight monoid $S \subseteq K$. We say that the K -grading of R is *pointed* if the weight cone $\omega \subseteq K_{\mathbb{Q}}$ contains no line and $R_0 = \mathbb{K}$ holds. By an *automorphism* of the K -graded algebra R we mean a pair (ψ, F) , where $\psi: R \rightarrow R$ is an isomorphism of \mathbb{K} -algebras and $F: K \rightarrow K$ is an isomorphism such that $\psi(R_w) = R_{F(w)}$ holds for all $w \in K$. We denote the group of these automorphisms of R by $\mathrm{Aut}(R, K)$.

Proposition 2.2. *Let K be a finitely generated abelian group and $R = \bigoplus_{w \in K} R_w$ a finitely generated integral \mathbb{K} -algebra with $R^* = \mathbb{K}^*$. Suppose that the grading is pointed. Then $\text{Aut}(R, K)$ is a linear algebraic group over \mathbb{K} and R is a rational $\text{Aut}(R, K)$ -module.*

Proof. The idea is to represent the automorphism group $\text{Aut}(R, K)$ as a closed subgroup of the linear automorphism group of a suitable finite dimensional vector subspace $V^0 \subseteq R$. In the subsequent construction of V^0 , we may assume that the weight cone ω generates $K_{\mathbb{Q}}$ as a vector space.

Consider the subgroup $\Gamma \subseteq \text{Aut}(K)$ of \mathbb{Z} -module automorphisms $K \rightarrow K$ such that the induced linear isomorphism $K_{\mathbb{Q}} \rightarrow K_{\mathbb{Q}}$ leaves the weight cone $\omega \subseteq K_{\mathbb{Q}}$ invariant. By finite generation of R , the cone ω is polyhedral and thus Γ is finite. Let $f_1, \dots, f_r \in R$ be homogeneous generators and denote by $w_i := \deg(f_i) \in K$ their degrees. Define a finite Γ -invariant subset and a vector subspace

$$S^0 := \Gamma \cdot \{w_1, \dots, w_r\} \subseteq K, \quad V^0 := \bigoplus_{w \in S^0} R_w \subseteq R.$$

For every automorphism (ψ, F) of the graded algebra R , we have $F(S^0) = S^0$ and thus $\psi(V^0) = V^0$. Moreover, (ψ, F) is uniquely determined by its restriction on V^0 . Consequently, we may regard the automorphism group $H := \text{Aut}(R, K)$ as a subgroup of the general linear group $\text{GL}(V^0)$. Note that every $g \in H$

- (i) permutes the components R_w of the decomposition $V^0 = \bigoplus_{w \in S^0} R_w$,
- (ii) satisfies $\sum_{\nu} a_{\nu} g(f_1)^{\nu_1} \cdots g(f_r)^{\nu_r} = 0$ for any relation $\sum_{\nu} a_{\nu} f_1^{\nu_1} \cdots f_r^{\nu_r} = 0$.

Obviously, these are algebraic conditions. Moreover, every $g \in \text{GL}(V^0)$ satisfying the above conditions can be extended uniquely to an element of $\text{Aut}(R, K)$ via

$$g \left(\sum_{\nu} a_{\nu} f_1^{\nu_1} \cdots f_r^{\nu_r} \right) := \sum_{\nu} a_{\nu} g(f_1)^{\nu_1} \cdots g(f_r)^{\nu_r}.$$

Thus, we saw that $H \subseteq \text{GL}(V^0)$ is precisely the closed subgroup defined by the above conditions (i) and (ii). In particular $H = \text{Aut}(R, K)$ is linear algebraic. Moreover, the symmetric algebra SV^0 is a rational $\text{GL}(V^0)$ -module, hence SV^0 is a rational H -module for the algebraic subgroup H of $\text{GL}(V^0)$, and so is its factor module R . \square

Corollary 2.3. *Let K be a finitely generated abelian group and $R = \bigoplus_{w \in K} R_w$ a finitely generated integral \mathbb{K} -algebra with $R^* = \mathbb{K}^*$ and consider the corresponding action of $H := \text{Spec } \mathbb{K}[K]$ on $\overline{X} := \text{Spec } R$. Then we have a canonical isomorphism*

$$\text{Aut}(\overline{X}, H) \rightarrow \text{Aut}(R, K), \quad (\varphi, \tilde{\varphi}) \mapsto (\varphi^*, \tilde{\varphi}^*),$$

where φ^* is the pullback of regular functions and $\tilde{\varphi}^*$ the pullback of characters. If the K -grading is pointed, then $\text{Aut}(\overline{X}, H)$ is a linear algebraic group acting morphically on \overline{X} .

We will also need details of the construction of the Cox sheaf \mathcal{R} on X which we briefly recall now. Denote by $c: \text{WDiv}(X) \rightarrow \text{Cl}(X)$ the map sending the Weil divisors to their classes, let $\text{PDiv}(X) = \ker(c)$ denote the group of principal divisors and choose a character, i.e. a group homomorphism $\chi: \text{PDiv}(X) \rightarrow \mathbb{K}(X)^*$ with

$$\text{div}(\chi(E)) = E, \quad \text{for all } E \in \text{PDiv}(X).$$

This can be done by prescribing χ suitably on a \mathbb{Z} -basis of $\text{PDiv}(X)$; see [13] for the existence of such a basis. Consider the associated sheaf of divisorial algebras

$$\mathcal{S} := \bigoplus_{\text{WDiv}(X)} \mathcal{S}_D, \quad \mathcal{S}_D := \mathcal{O}_X(D).$$

Denote by \mathcal{I} the sheaf of ideals of \mathcal{S} locally generated by the sections $1 - \chi(E)$, where 1 is homogeneous of degree zero, E runs through $\text{PDiv}(X)$ and $\chi(E)$ is

homogeneous of degree $-E$. The *Cox sheaf* associated to K and χ is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\text{Cl}(X)$ -grading

$$\mathcal{R} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}, \quad \mathcal{R}_{[D]} := \pi \left(\bigoplus_{D' \in c^{-1}([D])} \mathcal{S}_{D'} \right).$$

where $\pi: \mathcal{S} \rightarrow \mathcal{R}$ denotes the projection. The Cox sheaf \mathcal{R} is a quasicoherent sheaf of $\text{Cl}(X)$ -graded \mathcal{O}_X -algebras. The *Cox ring* is the ring $\mathcal{R}(X)$ of global sections of the Cox sheaf.

Proof of Theorem 2.1. We set $G_{\overline{X}} := \text{Aut}(\overline{X}, H_X)$ for short. According to Corollary 2.3, the group $G_{\overline{X}}$ is linear algebraic and acts morphically on \overline{X} . Looking at the representations of H_X and $G_{\overline{X}}$ on $\Gamma(\overline{X}, \mathcal{O}) = \mathcal{R}(X)$ defined by the respective actions, we see that the canonical inclusion $H_X \rightarrow G_{\overline{X}}$ is a morphism of linear algebraic groups.

Next we construct the subset $U \subseteq X$ from the last part of the statement. Consider the translates $g \cdot \widehat{X}$, where $g \in G_{\overline{X}}$. Each of them admits a good quotient with a complete quotient space:

$$p_{X,g}: g \cdot \widehat{X} \rightarrow (g \cdot \widehat{X}) // H_X.$$

By [3], there are only finitely many open subsets of \overline{X} with such a good quotient. In particular, the number of translates $g \cdot \widehat{X}$ is finite.

Let $W \subseteq X$ denote the maximal open subset such that the restricted quotient $\widehat{W} \rightarrow W$, where $\widehat{W} := p_X^{-1}(W)$, is geometric, i.e. has the H_X -orbits as its fibers. Then, for any $g \in G_{\overline{X}}$, the translate $g \cdot \widehat{W} \subseteq g \cdot \widehat{X}$ is the (unique) maximal open subset which is saturated with respect to the quotient map $p_{X,g}$ and defines a geometric quotient. Consider

$$\widehat{U} := \bigcap_{g \in G_{\overline{X}}} g \cdot \widehat{W} \subseteq \widehat{X}.$$

By the preceding considerations \widehat{U} is open, and by construction it is $G_{\overline{X}}$ -invariant and saturated with respect to p_X . By [1, Prop. 6.1.6] the set \widehat{W} is big in \overline{X} . Consequently, also \widehat{U} is big in \overline{X} . Thus, the (open) set $U := p_X(\widehat{U})$ is big in X . By the universal property of the geometric quotient, there is a unique morphical action of $G_{\overline{X}}$ on U making $p_X: \widehat{U} \rightarrow U$ equivariant. Thus, we have homomorphism of groups

$$G_{\overline{X}} \rightarrow \text{Aut}(U) \subseteq \text{Bir}_2(X).$$

We show that $G_{\overline{X}} \rightarrow \text{Bir}_2(X)$ is surjective. Consider a weak automorphism $\varphi: X \rightarrow X$. The pullback defines an automorphism of the group of Weil divisors

$$\varphi^*: \text{WDiv}(X) \rightarrow \text{WDiv}(X), \quad D \mapsto \varphi^* D.$$

As in the construction of the Cox sheaf, consider the sheaf of divisorial algebras $\mathcal{S} = \bigoplus \mathcal{S}_D$ associated to $\text{WDiv}(X)$ and fix a character $\chi: \text{PDiv}(X) \rightarrow \mathbb{K}(X)^*$ with $\text{div}(\chi(E)) = E$ for any $E \in \text{PDiv}(X)$. Then we obtain a homomorphism

$$\alpha: \text{PDiv}(X) \rightarrow \mathbb{K}^*, \quad E \mapsto \frac{\varphi^*(\chi(E))}{\chi(\varphi^*(E))}.$$

We extend this to a homomorphism $\alpha: \text{WDiv}(X) \rightarrow \mathbb{K}^*$ as follows. Write $\text{Cl}(X)$ as a direct sum of a free part and cyclic groups $\Gamma_1, \dots, \Gamma_s$ of order n_i . Take $D_1, \dots, D_r \in \text{WDiv}(X)$ such that the classes of D_1, \dots, D_s are generators for $\Gamma_1, \dots, \Gamma_s$ and the remaining ones define a basis of the free part. Set

$$\alpha(D_i) := \sqrt[n_i]{\alpha(n_i D_i)} \text{ for } 1 \leq i \leq s, \quad \alpha(D_i) := 1 \text{ for } s+1 \leq i \leq r.$$

Then one directly checks that this extends α to a homomorphism $\mathrm{WDiv}(X) \rightarrow \mathbb{K}^*$. Using $\alpha(E)$ as a “correction term”, we define an automorphism of the graded sheaf \mathcal{S} of divisorial algebras: for any open set $V \subseteq X$ we set

$$\varphi^*: \Gamma(V, \mathcal{S}_D) \rightarrow \Gamma(\varphi^{-1}(V), \mathcal{S}_{\varphi^*(D)}), \quad f \mapsto \alpha(D)f \circ \varphi.$$

By construction φ^* sends the ideal \mathcal{I} arising from the character χ to itself. Consequently, φ^* descends to an automorphism (ψ, F) of the (graded) Cox sheaf \mathcal{R} ; note that F is the pullback of divisor classes via φ . The degree zero part of ψ equals the usual pullback of regular functions on X via φ . Thus, the element in $\mathrm{Aut}(\overline{X}, H_X)$ defined by $\mathrm{Spec} \psi: \widehat{U} \rightarrow \widehat{U}$ maps to φ .

Clearly, H_X lies in the kernel of $\pi: G_{\overline{X}} \rightarrow \mathrm{Bir}_2(X)$. For the reverse inclusion, consider an element $g \in \ker(\pi)$. Then g is a pair $(\varphi, \tilde{\varphi})$ and, by the construction of π , we have a commutative diagram

$$\begin{array}{ccc} \widehat{U} & \xrightarrow{\varphi} & \widehat{U} \\ p_X \downarrow & & \downarrow p_X \\ U & \xrightarrow{\mathrm{id}} & U \end{array}$$

In particular, φ stabilizes all H_X -invariant divisors. It follows that the pullback φ^* on $\Gamma(\widehat{U}, \mathcal{O}) = \mathcal{R}(X)$ stabilizes the homogeneous components. Thus, for any homogeneous f of degree w , we have $\varphi^*(f) = \lambda(w)f$ with a homomorphism $\lambda: K \rightarrow \mathbb{K}^*$. Consequently $\varphi(x) = h \cdot x$ holds with an element $h \in H_X$. The statements concerning the upper sequence are verified.

Now, consider the lower sequence. Since \widehat{X} is big in \overline{X} , every automorphism of \widehat{X} extends to an automorphism of \overline{X} . We conclude that $\mathrm{Aut}(\widehat{X}, H_X)$ is the (closed) subgroup of $G_{\overline{X}}$ leaving the complement $\overline{X} \setminus \widehat{X}$ invariant. As seen before, the collection of translates $G_{\overline{X}} \cdot \widehat{X}$ is finite and thus the subgroup $\mathrm{Aut}(\widehat{X}, H_X)$ of $G_{\overline{X}}$ is of finite index. Moreover, lifting $\varphi \in \mathrm{Aut}(X)$ as before gives an element of $\mathrm{Aut}(\overline{X}, H_X)$ leaving \widehat{X} invariant. Thus, $\mathrm{Aut}(\widehat{X}, H_X) \rightarrow \mathrm{Aut}(X)$ is surjective with kernel H_X . By the universal property of the good quotient $\widehat{X} \rightarrow X$, the action of $\mathrm{Aut}(X)$ on X is morphical. \square

Corollary 2.4. *The automorphism group $\mathrm{Aut}(X)$ of a Mori dream space X is linear algebraic and acts morphically on X .*

Corollary 2.5. *If two Mori dream spaces are isomorphic in codimension two, then the unit components of their automorphism groups are isomorphic to each other.*

Let $\mathrm{CAut}(\overline{X}, H_X)$ denote the centralizer of H_X in the automorphism group $\mathrm{Aut}(\overline{X})$. Then $\mathrm{CAut}(\overline{X}, H_X)$ consists of all automorphisms $\varphi: \overline{X} \rightarrow \overline{X}$ satisfying

$$\varphi(t \cdot x) = t \cdot \varphi(x) \quad \text{for all } x \in \overline{X}, t \in H_X.$$

In particular, we have $\mathrm{CAut}(\overline{X}, H_X) \subseteq \mathrm{Aut}(\overline{X}, H_X)$. The group $\mathrm{CAut}(\overline{X}, H_X)$ may be used to detect the unit component $\mathrm{Aut}(X)^0$ of the automorphism group of X .

Corollary 2.6. *Let X be a Mori dream space. Then there is an exact sequence of linear algebraic groups*

$$1 \longrightarrow H_X \longrightarrow \mathrm{CAut}(\overline{X}, H_X)^0 \longrightarrow \mathrm{Aut}(X)^0 \longrightarrow 1.$$

Proof. According to [14, Cor. 2.3], the group $\mathrm{CAut}(\overline{X}, H_X)^0$ leaves \widehat{X} invariant. Thus, we have $\mathrm{CAut}(\overline{X}, H_X)^0 \subseteq \mathrm{Aut}(\widehat{X}, H_X)$ and the sequence is well defined. Moreover, for any $\varphi \in \mathrm{Aut}(X)^0$, the pullback $\varphi^*: \mathrm{Cl}(X) \rightarrow \mathrm{Cl}(X)$ is the identity. Consequently, φ lifts to an element of $\mathrm{CAut}(\overline{X}, H_X)$. Exactness of the sequence thus follows by dimension reasons. \square

Corollary 2.7. *Let X be a Mori dream space. Then, for any closed subgroup $F \subseteq \text{Aut}(X)^0$, there is a closed subgroup $F' \subseteq \text{CAut}(\overline{X}, H_X)^0$ such that the induced map $F' \rightarrow F$ is an epimorphism with finite kernel.*

Corollary 2.8. *Let X be a Mori dream space such that the group $\text{CAut}(\overline{X}, H_X)$ is connected, e.g. a toric variety, then there is an exact sequence of linear algebraic groups*

$$1 \longrightarrow H_X \longrightarrow \text{CAut}(\overline{X}, H_X) \longrightarrow \text{Aut}(X)^0 \longrightarrow 1.$$

Example 2.9. Consider the nondegenerate quadric X in the projective space \mathbb{P}_{n+1} , where $n \geq 4$ is even. Then the Cox ring of X is the \mathbb{Z} -graded ring

$$\mathcal{R}(X) = \mathbb{K}[T_0, \dots, T_{n+1}] / \langle T_0^2 + \dots + T_{n+1}^2 \rangle, \quad \deg(T_0) = \dots = \deg(T_{n+1}) = 1.$$

The characteristic quasitorus is $H_X = \mathbb{K}^*$. Moreover, for the equivariant automorphisms and the centralizer of H_X we obtain

$$\text{Aut}(\overline{X}, H_X) = \text{CAut}(\overline{X}, H_X) = \mathbb{K}^* E_{n+2} \cdot O_{n+2}.$$

Thus, $\text{CAut}(\overline{X}, H_X)$ has two connected components. Note that for $n = 4$, the quadric X comes with a torus action of complexity one.

3. RINGS WITH A FACTORIAL GRADING OF COMPLEXITY ONE

Here we recall the necessary constructions and results on factorially graded rings of complexity one and Cox rings of varieties with a torus action of complexity one from [9]. The main result of this section is Proposition 3.5 which describes the dimension of the homogeneous components in terms of the (common) degree of the relations. As before, we work over an algebraically closed field \mathbb{K} of characteristic zero.

Let K be an abelian group and $R = \bigoplus_K R_w$ a K -graded algebra. The grading is called *effective* if the weight monoid S of R generates K as a group. Moreover, we say that the grading is of *complexity one*, if it is effective and $\dim(K_{\mathbb{Q}})$ equals $\dim(R) - 1$. By a *K -prime* element of R we mean a homogeneous nonzero nonunit $f \in R$ such that $f \mid gh$ with homogeneous $g, h \in R$ implies $f \mid g$ or $f \mid h$. We say that R is *factorially K -graded* if every nonzero homogeneous nonunit of R is a product of K -primes.

Construction 3.1. See [9, Section 1]. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \dots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \dots + n_r$ and let $m \in \mathbb{Z}_{\geq 0}$. The input data are

- a matrix $A := [a_0, \dots, a_r]$ with pairwise linearly independent column vectors $a_0, \dots, a_r \in \mathbb{K}^2$,
- an integral $r \times (n + m)$ block matrix $P_0 = (L_0, 0)$, where L_0 is a $r \times n$ matrix build from tuples $l_i := (l_{i1}, \dots, l_{in_i}) \in \mathbb{Z}_{\geq 1}^{n_i}$ as follows

$$L_0 = \begin{bmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{bmatrix}.$$

Consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ in the variables T_{ij} , where $0 \leq i \leq r$, $1 \leq j \leq n_i$ and S_k , where $1 \leq k \leq m$. For every $0 \leq i \leq r$, define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}}.$$

Denote by \mathcal{I} the set of all triples $I = (i_1, i_2, i_3)$ with $0 \leq i_1 < i_2 < i_3 \leq r$ and define for any $I \in \mathcal{I}$ a trinomial

$$g_I := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}.$$

Let P_0^* denote the transpose of P_0 . We introduce a grading on $\mathbb{K}[T_{ij}, S_k]$ by the factor group $K_0 := \mathbb{Z}^{n+m}/\text{im}(P_0^*)$. Let $Q_0: \mathbb{Z}^{n+m} \rightarrow K_0$ be the projection and set

$$\deg(T_{ij}) := w_{ij} := Q_0(e_{ij}), \quad \deg(T_k) := w_k := Q_0(e_k),$$

where $e_{ij} \in \mathbb{Z}^{n+m}$, for $0 \leq i \leq r$, $1 \leq j \leq n_i$, and $e_k \in \mathbb{Z}^{n+m}$, for $1 \leq k \leq m$, are the canonical basis vectors. Note that all the g_I are K_0 -homogeneous of degree

$$\mu := l_{01}w_{01} + \dots + l_{0n_0}w_{0n_0} = \dots = l_{r1}w_{r1} + \dots + l_{rn_r}w_{rn_r} \in K_0.$$

In particular, the trinomials g_I generate a K_0 -homogeneous ideal and thus we obtain a K_0 -graded factor algebra

$$R(A, P_0) := \mathbb{K}[T_{ij}, S_k] / \langle g_I; I \in \mathfrak{I} \rangle.$$

Theorem 3.2. *See [9, Theorems 1.1 and 1.3]. With the notation of Construction 3.1, the following statements hold.*

- (i) *The K_0 -grading of ring $R(A, P_0)$ is effective, pointed, factorial and of complexity one.*
- (ii) *The variables T_{ij} and S_k define a system of pairwise nonassociated K_0 -prime generators of $R(A, P_0)$.*
- (iii) *Every finitely generated normal \mathbb{K} -algebra with an effective, pointed, factorial grading of complexity one is isomorphic to some $R(A, P_0)$.*

Note that in the case $r = 1$, there are no relations and the theorem thus treats the effective, pointed gradings of complexity one of the polynomial ring.

Example 3.3 (The E_6 -singular cubic I). Let $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 0$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P_0 = L_0 = \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \end{bmatrix}.$$

Then we have exactly one triple in \mathfrak{I} , namely $I = (1, 2, 3)$, and, as a ring, $R(A, P_0)$ is given by

$$R(A, P_0) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] / \langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2 \rangle.$$

The grading group $K_0 = \mathbb{Z}^4/\text{im}(P_0^*)$ is isomorphic to \mathbb{Z}^2 and the grading can be given explicitly via

$$\begin{aligned} \deg(T_{01}) &= \begin{pmatrix} -3 \\ 3 \end{pmatrix}, & \deg(T_{02}) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \deg(T_{11}) &= \begin{pmatrix} 0 \\ 2 \end{pmatrix}, & \deg(T_{21}) &= \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \end{aligned}$$

Recall that for any integral ring $R = \bigoplus_K R_w$ graded by an abelian group K , one has the subfield of degree zero fractions inside the field of fractions:

$$Q(R)_0 = \left\{ \frac{f}{g}; f, g \in R \text{ homogeneous}, g \neq 0, \deg(f) = \deg(g) \right\} \subseteq Q(R).$$

Proposition 3.4. *Take any i, j with $i \neq j$ and $0 \leq i, j \leq r$. Then the field of degree zero fractions of the ring $R(A, P_0)$ is the rational function field*

$$Q(R(A, P_0))_0 = \mathbb{K} \left(\frac{T_j^{l_j}}{T_i^{l_i}} \right).$$

Proof. It suffices to treat the case $m = 0$. Let $F = \prod T_{ij}$ be the product of all variables. Then $\mathbb{T}^n = \mathbb{K}_F^n$ is the n -torus and P_0 defines an epimorphism having the quasitorus $H_0 := \text{Spec } \mathbb{K}[K_0]$ as its kernel

$$\pi: \mathbb{T}^n \rightarrow \mathbb{T}^r, \quad (t_{ij}) \mapsto \left(\frac{t_1^{l_1}}{t_0^{l_0}}, \dots, \frac{t_r^{l_r}}{t_0^{l_0}} \right).$$

Set $\overline{X} := \text{Spec } R(A, P_0)$. Then $\pi(\overline{X}_F) = \overline{X}_F/H_0$ is a curve defined by affine linear equations in the coordinates of \mathbb{T}^r and thus rational. The assertion follows. \square

The following observation shows that the common degree $\mu = \deg(g_I)$ of the relations generalizes the “remarkable weight” introduced by Panyushev [12] in the factorial case. Recall that the weight monoid $S_0 \subseteq K_0$ consists of all $w \in K_0$ admitting a nonzero homogeneous element.

Proposition 3.5. *Consider the K_0 -graded ring $R := R(A, P_0)$ and the degree $\mu = \deg(g_I)$ of the relations as defined in 3.1. For $w \in S_0$ let $s_w \in \mathbb{Z}_{\geq 0}$ be the unique number with $w - s_w\mu \in S_0$ and $w - (s_w + 1)\mu \notin S_0$. Then we have*

$$\dim(R_w) = s_w + 1 \text{ for all } w \in S_0.$$

The element $\mu \in K_0$ is uniquely determined by this property. We have $\dim(R_\mu) = 2$ and any two nonproportional elements in R_μ are coprime. Moreover, any $w \in S_0$ with $w - \mu \notin S_0$ satisfies $\dim(R_w) = 1$.

Proof. According to Proposition 3.4, the field $Q(R)_0$ of degree zero fractions is the field of rational functions in p_1/p_0 , where $p_0 := T_0^{l_0}$ and $p_1 := T_1^{l_1}$ are coprime and of degree μ . Moreover, by the structure of the relations g_I , we have $\dim(R_\mu) = 2$.

Now, consider $w \in S_0$. If we have $\dim(R_w) = 1$, then $\dim(R_\mu) = 2$ implies $s_w = 0$ and the assertion follows in this case. Suppose that we have $\dim(R_w) > 1$. Then we find two nonproportional elements $f_0, f_1 \in R_w$ and two coprime homogeneous polynomials F_0, F_1 of a common degree $s > 0$ such that

$$\frac{f_1}{f_0} = \frac{F_1(p_0, p_1)}{F_0(p_0, p_1)}.$$

Observe that $F_1(p_0, p_1)$ must divide f_1 . This implies $w - s\mu \in S_0$. Repeating the procedure with $w - s\mu$ and so on, we finally arrive at a weight $\tilde{w} = w - s_w\mu$ with $\dim(R_{\tilde{w}}) = 1$. Moreover, by the procedure, any element of R_w is of the form $hF(p_0, p_1)$ with $0 \neq h \in R_{\tilde{w}}$ and a homogeneous polynomial F of degree s_w . The assertion follows. \square

Corollary 3.6. *If we have $r \geq 2$ and $l_{i1} + \dots + l_{in_i} \geq 2$ holds for all i , then the homogeneous components $R_{w_{ij}}, R_{w_k}$ of the generators T_{ij}, S_k of the K_0 -graded ring $R := R(A, P_0)$ are all of dimension one.*

We turn to Cox rings of varieties with a complexity one torus action. They are obtained by suitably downgrading the rings $R(A, P_0)$ as follows.

Construction 3.7. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \dots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \dots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data are

- a matrix $A := [a_0, \dots, a_r]$ with pairwise linearly independent column vectors $a_0, \dots, a_r \in \mathbb{K}^2$,
- an integral block matrix P of size $(r + s) \times (n + m)$ the columns of which are pairwise different primitive vectors generating \mathbb{Q}^{r+s} as a cone:

$$P = \begin{pmatrix} L_0 & 0 \\ d & d' \end{pmatrix},$$

where d is an $(s \times n)$ -matrix, d' an $(s \times m)$ -matrix and L_0 an $(r \times n)$ -matrix build from tuples $l_i := (l_{i1}, \dots, l_{in_i}) \in \mathbb{Z}_{\geq 1}^{n_i}$ as in 3.1.

Let P^* denote the transpose of P , consider the factor group $K := \mathbb{Z}^{n+m}/\text{im}(P^*)$ and the projection $Q: \mathbb{Z}^{n+m} \rightarrow K$. We define a K -grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := Q(e_{ij}), \quad \deg(S_k) := Q(e_k).$$

The trinomials g_I of 3.1 are K -homogeneous, all of the same degree. In particular, we obtain a K -graded factor ring

$$R(A, P) := \mathbb{K}[T_{ij}, S_k; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m] / \langle g_I; I \in \mathfrak{I} \rangle.$$

Theorem 3.8. *See [9, Theorem 1.4]. With the notation of Construction 3.7, the following statements hold.*

- (i) *The K -grading of the ring $R(A, P)$ is factorial, pointed and almost free, i.e. K is generated by any $n - m - 1$ of the $\deg(T_{ij}), \deg(T_k)$.*
- (ii) *The variables T_{ij} and S_k define a system of pairwise nonassociated K -prime generators of $R(A, P)$.*

Remark 3.9. As rings $R(A, P_0)$ and $R(A, P)$ coincide but the K_0 -grading is finer than the K -grading. The downgrading map $K_0 \rightarrow K$ fits into the following commutative diagram built from exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow \mathbb{Z}^s & & \\
 0 & \longrightarrow & \mathbb{Z}^r & \xrightarrow{P_0^*} & \mathbb{Z}^{n+m} & \xrightarrow{Q_0} & K_0 \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & \mathbb{Z}^{r+s} & \xrightarrow{P^*} & \mathbb{Z}^{n+m} & \xrightarrow{Q} & K \longrightarrow 0 \\
 & & \downarrow \mathbb{Z}^s & & & & \downarrow 0 \\
 & & 0 & & & &
 \end{array}$$

The snake lemma [11, Sec. III.9] allows us to identify the direct factor \mathbb{Z}^s of \mathbb{Z}^{r+s} with the kernel of the downgrading map $K_0 \rightarrow K$. Note that for the quasitori T , H_0 and H associated to abelian groups \mathbb{Z}^s , K_0 and K we have $T = H_0/H$.

Construction 3.10. Consider a ring $R(A, P)$ with its K -grading and the finer K_0 -grading. Then the quasitori $H = \text{Spec } \mathbb{K}[K]$ and $H_0 := \text{Spec } \mathbb{K}[K_0]$ act on $\overline{X} := \text{Spec } R(A, P)$. Let $\widehat{X} \subseteq \overline{X}$ be a big H_0 -invariant open subset with a good quotient

$$p: \widehat{X} \rightarrow X = \widehat{X} // H$$

such that X is complete and for some open set $U \subseteq X$, the inverse image $p^{-1}(U) \subseteq \widehat{X}$ is big and H acts freely on U . Then X is a Mori dream space of dimension $s + 1$ with divisor class group $\text{Cl}(X) \cong K$ and Cox ring $\mathcal{R}(X) \cong R(A, P)$. Moreover, X comes with an induced effective action of the s -dimensional torus $T := H_0/H$.

Theorem 3.11. *Let X be an n -dimensional complete normal rational variety with an effective action of an $(n - 1)$ -dimensional torus S . Then X is equivariantly isomorphic to a T -variety arising from data (A, P) as in 3.10.*

Proof. We may assume that X is not a toric variety. According to [9, Theorem 1.5], the $\text{Cl}(X)$ -graded Cox ring of X is isomorphic to a K -graded ring $R(A, P)$. Thus, in the notation of 3.10, there is a big H -invariant open subset \widehat{X} of \overline{X} with $X \cong \widehat{X} // H$. Applying [14, Cor. 2.3] to a subtorus $T_0 \subseteq H_0$ projecting onto $T = H_0/H$, we see that \widehat{X} is even invariant under H_0 . Thus, T acts on X . Since the T -action is conjugate in $\text{Aut}(X)$ to the given S -action on X , the assertion follows. \square

Example 3.12 (The E_6 -singular cubic II). Let $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 0$, $s = 1$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}.$$

Then, as remarked before, we have exactly one triple $I = (1, 2, 3)$ and, as a ring, $R(A, P)$ is given by

$$R(A, P) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] / \langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2 \rangle.$$

The grading group $K = \mathbb{Z}^4 / \text{im}(P^*)$ is isomorphic to \mathbb{Z} and the grading can be given explicitly via

$$\deg(T_{01}) = 3, \quad \deg(T_{02}) = 1, \quad \deg(T_{11}) = 2, \quad \deg(T_{21}) = 3.$$

As shown for example in [8, Example 3.7], the ring $R(A, P)$ is the Cox ring of the E_6 -singular cubic surface in the projective space given by

$$X = V(z_1 z_2^2 + z_2 z_0^2 + z_3^3) \subseteq \mathbb{P}_3.$$

4. PRIMITIVE LOCALLY NILPOTENT DERIVATIONS

Here, we investigate the homogeneous locally nilpotent derivations of the K_0 -graded algebra $R(A, P_0)$. The description of the “primitive” ones given in Theorem 4.3 is the central algebraic tool for our study of automorphism groups. As before, \mathbb{K} is an algebraically closed field of characteristic zero.

Let us briefly recall the necessary background. We consider derivations on an integral \mathbb{K} -algebra R , that means \mathbb{K} -linear maps $\delta: R \rightarrow R$ satisfying the Leibniz rule

$$\delta(fg) = \delta(f)g + f\delta(g).$$

Any such $\delta: R \rightarrow R$ extends uniquely to a derivation $\delta: Q(R) \rightarrow Q(R)$ of the quotient field. Recall that a derivation $\delta: R \rightarrow R$ is said to be *locally nilpotent* if for every $f \in R$ there is an $n \in \mathbb{N}$ with $\delta^n(f) = 0$. Now suppose that R is graded by a finitely generated abelian group:

$$R = \bigoplus_{w \in K} R_w.$$

A derivation $\delta: R \rightarrow R$ is called *homogeneous* if for every $w \in K$ there is a $w' \in K$ with $\delta(R_w) \subseteq R_{w'}$. Any homogeneous derivation $\delta: R \rightarrow R$ has a *degree* $\deg(\delta) \in K$ satisfying $\delta(R_w) \subseteq R_{w+\deg(\delta)}$ for all $w \in K$.

Definition 4.1. Let K be a finitely generated abelian group, $R = \bigoplus_K R_w$ a K -graded \mathbb{K} -algebra and $Q(R)_0 \subseteq Q(R)$ the subfield of all fractions f/g of homogeneous elements $f, g \in R$ with $\deg(f) = \deg(g)$.

- (i) We call a homogeneous derivation $\delta: R \rightarrow R$ *primitive* if $\deg(\delta)$ does not lie in the weight cone $\omega \subseteq K_{\mathbb{Q}}$ of R .
- (ii) We say that a homogeneous derivation $\delta: R \rightarrow R$ is of *vertical type* if $\delta(Q(R)_0) = 0$ holds and of *horizontal type* otherwise.

Construction 4.2. Notation as in Construction 3.1. We define derivations of the K_0 -graded algebra $R(A, P_0)$ constructed there. The input data are

- a sequence $C = (c_0, \dots, c_r)$ with $1 \leq c_i \leq n_i$,
- a vector $\beta \in \mathbb{K}^{r+1}$ lying in the row space of the matrix $[a_0, \dots, a_r]$.

Note that for $0 \neq \beta$ as above either all entries differ from zero or there is a unique i_0 with $\beta_{i_0} = 0$. According to these cases, we put further conditions and define:

- (i) if all entries β_0, \dots, β_r differ from zero and there is at most one i_1 with $l_{i_1 c_{i_1}} > 1$, then we set

$$\begin{aligned} \delta_{C, \beta}(T_{ij}) &:= \begin{cases} \beta_i \prod_{k \neq i} \frac{\partial T_k^{l_k}}{\partial T_{k c_k}}, & j = c_i, \\ 0, & j \neq c_i, \end{cases} \\ \delta_{C, \beta}(S_k) &:= 0 \quad \text{for } k = 1, \dots, m, \end{aligned}$$

- (ii) if $\beta_{i_0} = 0$ is the unique zero entry of β and there is at most one i_1 with $i_1 \neq i_0$ and $l_{i_1 c_{i_1}} > 1$, then we set

$$\begin{aligned} \delta_{C,\beta}(T_{ij}) &:= \begin{cases} \beta_i \prod_{k \neq i, i_0} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}}, & j = c_i, \\ 0, & j \neq c_i, \end{cases} \\ \delta_{C,\beta}(S_k) &:= 0 \quad \text{for } k = 1, \dots, m. \end{aligned}$$

These assignments define K_0 -homogeneous primitive locally nilpotent derivations $\delta_{C,\beta}: R(A, P_0) \rightarrow R(A, P_0)$ of degree

$$\deg(\delta_{C,\beta}) = \begin{cases} r\mu - \sum_k \deg(T_{kc_k}), & \text{in case (i),} \\ (r-1)\mu - \sum_{k \neq i_0} \deg(T_{kc_k}), & \text{in case (ii).} \end{cases}$$

Proof. The assignments (i) and (ii) on the variables define a priori derivations of the polynomial ring $\mathbb{K}[T_{ij}, S_k]$. Recall from 3.1 that $R(A, P_0)$ is the quotient of $\mathbb{K}[T_{ij}, S_k]$ by the ideal generated by all

$$g_I = \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix},$$

where $I = (i_1, i_2, i_3)$. Since the vector β lies in the row space of $[a_0, \dots, a_r]$, we see that $\delta_{C,\beta}$ sends every trinomial g_I to zero and thus descends to a well defined derivation of $R(A, P_0)$.

We check that $\delta_{C,\beta}$ is homogeneous. Obviously, every $\delta_{C,\beta}(T_{ij})$ is a K_0 -homogeneous element of $\mathbb{K}[T_{ij}]$. Moreover, with the degree μ of the relations g_I , we have

$$\deg(\delta_{C,\beta}(T_{ij})) - \deg(T_{ij}) = \begin{cases} r\mu - \sum_k \deg(T_{kc_k}), & \text{in case (i),} \\ (r-1)\mu - \sum_{k \neq i_0} \deg(T_{kc_k}), & \text{in case (ii).} \end{cases}$$

In particular, the left hand side does not depend on (i, j) . We conclude that $\delta_{C,\beta}$ is homogeneous of degree $\deg(\delta_{C,\beta}(T_{ij})) - \deg(T_{ij})$.

For primitivity, we have to show that the degree of $\delta_{C,\beta}$ does not lie in the weight cone of $R(A, P_0)$. We exemplarily treat case (i). As seen before, the degree of $\delta_{C,\beta}$ is represented by the vector

$$v_{C,\beta} := -e_{0c_0} + \sum_{j \neq c_1} l_{1j} e_{1j} + \dots + \sum_{j \neq c_r} l_{rj} e_{rj} \in \mathbb{Z}^n.$$

Thus, we look for a linear form on \mathbb{Q}^n separating this vector from the orthant cone(e_{ij}) and vanishing along the kernel of $\mathbb{Q}^n \rightarrow (K_0)_{\mathbb{Q}}$, i.e. the linear subspace spanned by the columns of P_0^* . For example, we may take

$$l_{0c_0}^{-1} e_{0c_0}^* + l_{1c_1}^{-1} e_{1c_1}^* + \dots + l_{rc_r}^{-1} e_{rc_r}^*.$$

Finally, we show that $\delta_{C,\beta}$ is locally nilpotent. If $l_{ic_i} = 1$ holds for all i , then $\delta_{C,\beta}(T_{ij})$ is a product of variables from the kernel of $\delta_{C,\beta}$ and thus $\delta_{C,\beta}^2$ annihilates all generators. If $l_{0c_0} > 1$ holds, then we have

$$\delta_{C,\beta}^2(T_{0c_0}) = 0, \quad \delta_{C,\beta}(T_{ic_i}) = T_{0c_0}^{l_{0c_0}-1} h_i,$$

where $i \geq 1$ and h_i lies in the kernel of $\delta_{C,\beta}$. Putting all together, we obtain that $\delta_{C,\beta}^{l_{0c_0}+1}$ annihilates all generators T_{ij} and thus $\delta_{C,\beta}$ is locally nilpotent. \square

Theorem 4.3. *Let $\delta: R(A, P_0) \rightarrow R(A, P_0)$ be a nontrivial primitive K_0 -homogeneous locally nilpotent derivation.*

- (i) *If δ is of vertical type, then $\delta(T_{ij}) = 0$ holds for all i, j and there is a k_0 such that $\delta(S_{k_0})$ does not depend on S_{k_0} and $\delta(S_k) = 0$ holds for all $k \neq k_0$.*
- (ii) *If δ is of horizontal type, then we have $\delta = h\delta_{C,\beta}$, where $\delta_{C,\beta}$ is as in 4.2 and $h \in \text{Ker}(\delta_{C,\beta})$ holds.*

In the proof of this theorem we will make frequently use of the following facts; the statements of the first Lemma occur in Freudenburg's book, see [6, Principles 1, 5 and 7, Corollary 1.20].

Lemma 4.4. *Let R be an integral \mathbb{K} -algebra, $\delta: R \rightarrow R$ a locally nilpotent derivation and let $f, g \in R$.*

- (i) *If $fg \in \text{Ker}(\delta)$ holds, then $f, g \in \text{Ker}(\delta)$ holds.*
- (ii) *If $\delta(f) = fg$ holds, then $\delta(f) = 0$ holds.*
- (iii) *The derivation $f\delta$ is locally nilpotent if and only if $f \in \text{Ker}(\delta)$ holds.*
- (iv) *If $g \mid \delta(f)$ and $f \mid \delta(g)$, then $\delta(f) = 0$ or $\delta(g) = 0$.*

Lemma 4.5. *Let $\delta: R(A, P_0) \rightarrow R(A, P_0)$ be a primitive K_0 -homogeneous derivation induced by a K_0 -homogeneous derivation $\widehat{\delta}: \mathbb{K}[T_{ij}, S_k] \rightarrow \mathbb{K}[T_{ij}, S_k]$. Then $\widehat{\delta}(g_I) = 0$ holds for all relations g_I .*

Proof. Clearly, we have $\widehat{\delta}(\mathfrak{a}) \subseteq \mathfrak{a}$ for the ideal $\mathfrak{a} \subseteq \mathbb{K}[T_{ij}, S_k]$ generated by the g_I . Recall that all g_I are of the same degree μ . By primitivity, $\deg(\delta) = \deg(\widehat{\delta})$ is not in the weight cone. Thus, $\mathbb{K}[T_{ij}]_{\mu+\deg(\delta)} \cap \mathfrak{a} = \{0\}$ holds. This implies $\widehat{\delta}(g_I) = 0$. \square

Proof of Theorem 4.3. Suppose that δ is of vertical type. Then $\delta(T_i^{l_i}/T_s^{l_s}) = 0$ holds for any two $0 \leq i < s \leq r$. By the Leibniz rule, this implies

$$\delta(T_i^{l_i})T_s^{l_s} = T_i^{l_i}\delta(T_s^{l_s}).$$

We conclude that $T_i^{l_i}$ divides $\delta(T_s^{l_s})$ and $T_s^{l_s}$ divides $\delta(T_i^{l_i})$. By Lemma 4.4 (ii), this implies $\delta(T_i^{l_i}) = \delta(T_s^{l_s}) = 0$. Using Lemma 4.4 (i), we obtain $\delta(T_{ij}) = 0$ for all variables T_{ij} . Since δ is nontrivial, we should have $\delta(S_{k_0}) \neq 0$ at least for one k_0 . Consider the basis $e_k = \deg(S_k)$ of \mathbb{Z}^m , where $k = 1, \dots, m$, and write

$$\deg(\delta) = w' + \sum_{k=1}^m b_k e_k, \quad \text{where } w' \in K_0 \quad \text{and} \quad b_k \in \mathbb{Z}.$$

Then $\deg(\delta(S_{k_0})) = w' + \sum_{k \neq k_0} b_k e_k + (b_{k_0} + 1)e_{k_0}$. By Lemma 4.4, the variable S_{k_0} does not divide $\delta(S_{k_0})$. This and the condition $\delta(S_{k_0}) \neq 0$ imply $b_{k_0} = -1$ and $b_k \geq 0$ for $k \neq k_0$. This proves that $\delta(S_k) = 0$ for all $k \neq k_0$ and $\delta(S_{k_0})$ is K_0 -homogeneous and does not depend on S_{k_0} .

Now suppose that δ is of horizontal type. Then there exists a variable T_{ij} with $\delta(T_{ij}) \neq 0$. Write

$$\deg(\delta(T_{ij})) = \deg(T_{ij}) + w' + \sum_{k=1}^m b_k e_k.$$

Then all coefficients b_k are nonnegative and consequently we obtain $\delta(S_k) = 0$ for $k = 1, \dots, m$.

We show that for any $T_i^{l_i}$ there is at most one variable T_{ij} with $\delta(T_{ij}) \neq 0$. Assume that we find two different j, k with $\delta(T_{ij}) \neq 0$ and $\delta(T_{ik}) \neq 0$. Note that we have

$$\frac{\partial T_i^{l_i}}{\partial T_{ij}} \delta(T_{ij}), \quad \frac{\partial T_i^{l_i}}{\partial T_{ik}} \delta(T_{ik}) \in R(A, P_0)_{\mu+\deg(\delta)}.$$

By Proposition 3.5, the component of degree $\mu + \deg(\delta)$ is of dimension one. Thus, the above two terms differ by a nontrivial scalar and we see that $T_{ik}^{l_{ik}}$ divides the second term. Consequently, T_{ik} must divide $\delta(T_{ik})$ which contradicts 4.4 (ii).

A second step is to see that for any two variables T_{ij} and T_{ks} with $\delta(T_{ij}) \neq 0$ and $\delta(T_{ks}) \neq 0$ we must have $l_{ij} = 1$ or $l_{ks} = 1$. Otherwise, we see as before that $\partial T_i^{l_i} / \partial T_{ij} \delta(T_{ij})$ and $\partial T_k^{l_k} / \partial T_{ks} \delta(T_{ks})$ differ by a nonzero scalar. Thus, we conclude $\delta(T_{ij}) = f T_{ks}$ and $\delta(T_{ks}) = h T_{ij}$, a contradiction to 4.4 (iv).

Finally, we prove the assertion. As already seen, for every $0 \leq k \leq r$ there is at most one c_k with $\delta(T_{kc_k}) \neq 0$. Let $\mathfrak{K} \subseteq \{0, \dots, r\}$ denote the set of all k admitting such a c_k . From Proposition 3.5 we infer $R(A, P)_{\mu + \deg(\delta)} = \mathbb{K}f$ with some nonzero element f . We claim that

$$f = h \prod_{k \in \mathfrak{K}} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}}, \quad \delta(T_{ic_i}) = \beta_i h \prod_{k \in \mathfrak{K} \setminus \{i\}} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}},$$

hold with a homogeneous element $h \in R(A, P)$ and scalars $\beta_0, \dots, \beta_r \in \mathbb{K}$. Indeed, similar to the previous arguments, the first equation follows from fact that all $\partial T_k^{l_k} / \partial T_{kc_k} \delta(T_{kc_k})$ are nonzero elements of the same degree as f and hence each $\partial T_k^{l_k} / \partial T_{kc_k}$ must divide f . The second equation is clear then.

The vector $\beta := (\beta_0, \dots, \beta_r)$ lies in the row space of the matrix A . To see this, consider the lift of δ to $\mathbb{K}[T_{ij}, S_k]$ defined by the second equation and apply Lemma 4.5. Now let $C = (c_0, \dots, c_r)$ be any sequence completing the c_k , where $k \in \mathfrak{K}$. Then we have $\delta = h\delta_{C, \beta}$. The fact that h belongs to the kernel of $\delta_{C, \beta}$ follows from Lemma 4.4. \square

Example 4.6 (The E_6 -singular cubic III). Situation as in 3.3. The locally nilpotent primitive homogeneous derivations of $R(A, P_0)$ of the form $\delta_{C, \beta}$ are the following

(i) $C = (1, 1, 1)$ and $\beta = (\beta_0, 0, -\beta_0)$. Here we have $\deg(\delta_{C, \beta}) = (3, 0)$ and

$$\delta_{C, \beta}(T_{01}) = 2\beta_0 T_{21}, \quad \delta_{C, \beta}(T_{21}) = -\beta_0 T_{02}^3, \quad \delta_{C, \beta}(T_{02}) = \delta_{C, \beta}(T_{11}) = 0.$$

(ii) $C = (1, 1, 1)$ and $\beta = (\beta_0, -\beta_0, 0)$. Here we have $\deg(\delta_{C, \beta}) = (3, 1)$ and

$$\delta_{C, \beta}(T_{01}) = 3\beta_0 T_{11}^2, \quad \delta_{C, \beta}(T_{11}) = -\beta_0 T_{02}^3, \quad \delta_{C, \beta}(T_{02}) = \delta_{C, \beta}(T_{21}) = 0.$$

The general locally nilpotent primitive homogeneous derivation δ of $R(A, P_0)$ has the form $h\delta_{C, \beta}$ with $h \in \text{Ker}(\delta_{C, \beta})$, and

$$\deg(\delta) = \deg(h) + \deg(\delta_{C, \beta}) \notin \omega.$$

In the above case (i), the only possibilities for $\deg(h)$ are $\deg(h) = (k, k)$ or $\deg(h) = (k, k) + (0, 2)$ and thus we have

$$\delta = T_{02}^k \delta_{C, \beta} \quad \text{or} \quad \delta = T_{02}^k T_{11} \delta_{C, \beta}.$$

In the above case (ii), the only possibility for $\deg(h)$ is $\deg(h) = (k, k)$ and thus we obtain

$$\delta = T_{02}^k \delta_{C, \beta}.$$

5. DEMAZURE ROOTS

Here we present and prove the main result, Theorem 5.4. It describes the root system of the automorphism group of a rational complete normal variety X coming with an effective torus action $T \times X \rightarrow X$ of complexity one in terms of the defining matrix P of the Cox ring $\mathcal{R}(X) = R(A, P)$, see Construction 3.10 and Theorem 3.11. We will assume that $R(A, P)$ is *minimally presented* in the sense that there occur no linear monomials in the defining relations g_I ; this can always be achieved by omitting redundant generators.

Definition 5.1. Let P be a matrix as in Construction 3.7. Denote by $v_{ij}, v_k \in N = \mathbb{Z}^{r+s}$ the columns of P and by M the dual lattice of N .

(i) A *vertical Demazure P -root* is a tuple (u, k_0) with a linear form $u \in M$ and an index $1 \leq k_0 \leq m$ satisfying

$$\begin{aligned} \langle u, v_{ij} \rangle &\geq 0 && \text{for all } i, j, \\ \langle u, v_k \rangle &\geq 0 && \text{for all } k \neq k_0, \\ \langle u, v_{k_0} \rangle &= -1. \end{aligned}$$

- (ii) A *horizontal Demazure P -root* is a tuple (u, i_0, i_1, C) , where $u \in M$ is a linear form, $i_0 \neq i_1$ are indices with $0 \leq i_0, i_1 \leq r$, and $C = (c_0, \dots, c_r)$ is a sequence with $1 \leq c_i \leq n_i$ such that

$$\begin{aligned} l_{ic_i} &= 1 && \text{for all } i \neq i_0, i_1, \\ \langle u, v_{ic_i} \rangle &= \begin{cases} 0, & i \neq i_0, i_1, \\ -1, & i = i_1, \end{cases} \\ \langle u, v_{ij} \rangle &\geq \begin{cases} l_{ij}, & i \neq i_0, i_1, \quad j \neq c_i, \\ 0, & i = i_0, i_1, \quad j \neq c_i, \\ 0, & i = i_0, \quad j = c_i, \end{cases} \\ \langle u, v_k \rangle &\geq 0 && \text{for all } k. \end{aligned}$$

We define the \mathbb{Z}^s -part of a Demazure P -root (u, k_0) or (u, i_0, i_1, C) to be the tuple of the last s coordinates of the linear form $u \in M = \mathbb{Z}^{r+s}$.

Example 5.2 (The E_6 -singular cubic IV). As earlier, let $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 0$, $s = 1$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}.$$

There are no vertical Demazure P -roots because of $m = 0$. There is a horizontal Demazure P -root $\kappa = (u, i_0, i_1, C)$ given by

$$u = (-1, -2, 3), \quad i_0 = 1, \quad i_1 = 2, \quad C = (1, 1, 1).$$

A direct computation shows that this is the only one. The \mathbb{Z}^s -part of κ is the third coordinate of the linear form u , i.e. it is $u_3 = 3 \in \mathbb{Z} = \mathbb{Z}^s$.

Note that the Demazure P -roots are certain Demazure roots [5, Section 3.1] of the fan with the rays through the columns of P as its maximal cones. In particular, there are only finitely many Demazure P -roots. For computing them explicitly, the following presentation is helpful.

Remark 5.3. The Demazure P -roots are the lattice points of certain polytopes in $M_{\mathbb{Q}}$. For an explicit description, we encode the defining conditions as a lattice vector $\zeta \in \mathbb{Z}^{n+m}$ and an affine subspace $\eta \subseteq M_{\mathbb{Q}}$:

- (i) For any index $1 \leq k_0 \leq m$ define a lattice vector $\zeta = (\zeta_{ij}, \zeta_k) \in \mathbb{Z}^{n+m}$ and an affine subspace $\eta \subseteq M_{\mathbb{Q}}$ by

$$\zeta_{ij} := 0 \text{ for all } i, j, \quad \zeta_k := 0 \text{ for all } k \neq k_0, \quad \zeta_{k_0} := -1,$$

$$\eta := \{u' \in M_{\mathbb{Q}}; \langle u', v_{k_0} \rangle = -1\} \subseteq M_{\mathbb{Q}}.$$

Then the vertical Demazure P -roots $\kappa = (u, k_0)$ are given by the lattice points u of the polytope

$$B(k_0) := \{u' \in \eta; P^*u' \geq \zeta\} \subseteq M_{\mathbb{Q}}.$$

- (ii) Given $i_0 \neq i_1$ with $0 \leq i_0, i_1 \leq r$ and $C = (c_0, \dots, c_r)$ with $1 \leq c_i \leq n_i$ such that $l_{ic_i} = 1$ holds for all $i \neq i_0, i_1$, set

$$\zeta_{ij} := \begin{cases} l_{ij}, & i \neq i_0, i_1, \quad j \neq c_i, \\ -1, & i = i_1, \quad j = c_{i_1}, \\ 0 & \text{else,} \end{cases} \quad \zeta_k = 0 \text{ for } 1 \leq k \leq m.$$

$$\eta := \{u' \in M_{\mathbb{Q}}; \langle u', v_{ic_i} \rangle = 0 \text{ for } i \neq i_0, i_1, \langle u', v_{i_1 c_{i_1}} \rangle = -1, \}.$$

Then the horizontal Demazure P -roots $\kappa = (u, i_0, i_1, C)$ are given by the lattice points u of the polytope

$$B(i_0, i_1, C) := \{u' \in \eta; P^*u' \geq \zeta\} \subseteq M_{\mathbb{Q}}.$$

In order to state and prove the main result, let us briefly recall the necessary concepts from the theory of linear algebraic groups G . One considers the adjoint representation of the torus T on the Lie algebra $\text{Lie}(G)$, i.e. the tangent representation at e_G of the T -action on G given by conjugation $(t, g) \mapsto tgt^{-1}$. There is a unique T -invariant splitting $\text{Lie}(G) = \text{Lie}(T) \oplus \mathfrak{n}$, where \mathfrak{n} is spanned by nilpotent vectors, and one has a bijection

$$1\text{-PASG}_T(G) \rightarrow \{T\text{-eigenvectors of } \mathfrak{n}\}, \quad \lambda \mapsto \dot{\lambda}(1).$$

Here $1\text{-PASG}_T(G)$ denotes the set of one parameter additive subgroups $\lambda: \mathbb{G}_a \rightarrow G$ normalized by T and $\dot{\lambda}$ denotes the differential. A root of G with respect to T is an eigenvalue of the T -representation on \mathfrak{n} , that means a character $\chi \in \mathbb{X}(T)$ with $t \cdot v = \chi(t)v$ for some T -eigenvector $0 \neq v \in \mathfrak{n}$.

Theorem 5.4. *Let X be a nontoric normal complete rational variety with minimally presented Cox ring $\mathcal{R}(X) = R(A, P)$, and a complexity one torus action $T \times X \rightarrow X$ as in Construction 3.10.*

- (i) *The automorphism group $\text{Aut}(X)$ is a linear algebraic group with maximal torus T .*
- (ii) *Under the canonical identification $\mathbb{X}(T) = \mathbb{Z}^s$, the roots of $\text{Aut}(X)$ with respect to T are the \mathbb{Z}^s -parts of the Demazure P -roots.*

The rest of the section is devoted to the proof. We will have to deal with the K_0 - and K -degrees of functions and derivations. It might be helpful to recall the relations between the gradings from Remark 3.9. The following simple facts will be frequently used.

Lemma 5.5. *In the setting of Constructions 3.1 and 3.7, consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ with the K_0 -grading and the coarser K -grading.*

- (i) *For a monomial $h = \prod T_{ij}^{e_{ij}} \prod S_k^{e_k}$ with exponent vector $e = (e_{ij}, e_k)$, the K_0 - and K -degrees are given as*

$$\deg_{K_0}(h) = Q_0(e), \quad \deg_K(h) = Q(e).$$

- (ii) *A monomial $h \in \mathbb{K}[T_{ij}, S_k]$ is of K -degree zero if and only if there is an $u \in M$ with*

$$h = h^u := \prod T_{ij}^{P^*(u)_{ij}} \prod S_k^{P^*(u)_k} = \prod T_{ij}^{\langle u, v_{ij} \rangle} \prod S_k^{\langle u, v_k \rangle}.$$

- (iii) *Let δ be a derivation on $\mathbb{K}[T_{ij}, S_k]$ sending the generators T_{ij}, S_k to monomials. The δ is K -homogeneous of K -degree zero if and only if*

$$\deg_K(T_{ij}^{-1}\delta(T_{ij})) = \deg_K(S_k^{-1}\delta(S_k)) = 0 \text{ holds for all } i, j, k.$$

If $0 \neq \delta$ is K_0 -homogeneous, then $\deg_K(\delta) = 0$ holds if and only if one of the $T_{ij}^{-1}\delta(T_{ij})$ and $S_k^{-1}\delta(S_k)$ is nontrivial of K -degree zero.

As a first step towards the roots of the automorphism group $\text{Aut}(X)$, we now associate K_0 -homogeneous locally nilpotent derivations of $R(A, P)$ to the Demazure P -roots.

Construction 5.6. Let A and P be as in Construction 3.7. For $u \in M$ and the lattice vector $\zeta \in \mathbb{Z}^{n+m}$ of Remark 5.3 consider the monomials

$$h^u = \prod_{i,j} T_{ij}^{\langle u, v_{ij} \rangle} \prod_k S_k^{\langle u, v_k \rangle}, \quad h^\zeta := \prod_{i,j} T_{ij}^{\zeta_{ij}} \prod_k S_k^{\zeta_k}.$$

According to the vertical and the horizontal case, we associate to any Demazure P -root κ a locally nilpotent derivation δ_κ of $R(A, P)$. If $\kappa = (u, k_0)$ is vertical, set

$$\delta_\kappa(T_{ij}) := 0 \text{ for all } i, j, \quad \delta_\kappa(S_k) := \begin{cases} S_{k_0} h^u, & k = k_0, \\ 0, & k \neq k_0. \end{cases}$$

If $\kappa = (u, i_0, i_1, C)$ is horizontal, then there is a unique vector β in the row space of A with $\beta_{i_0} = 0$, $\beta_{i_1} = 1$ and we set

$$\delta_\kappa := \frac{h^u}{h^\zeta} \delta_{C,\beta}.$$

In all cases, the derivation δ_κ is K_0 -homogeneous; its K_0 -degree is the \mathbb{Z}^s -part of κ and the K -degree is zero:

$$\deg_{K_0}(\delta_\kappa) = Q_0(P^*(u)), \quad \deg_K(\delta_\kappa) = 0.$$

Proof. In the vertical case $\delta_\kappa(S_{k_0})$ does not depend on S_{k_0} and in the horizontal case the factors before $\delta_{C,\beta}$ in the definitions of δ_κ are contained in $\text{Ker}(\delta_{C,\beta})$. Thus, the derivations δ_κ are locally nilpotent. Clearly, the δ_κ are K_0 -homogeneous. By Lemma 5.5, the monomial h^u is of K_0 -degree $Q_0(P^*(u))$. In the vertical case, this implies directly that δ_κ is of K_0 -degree $Q_0(P^*(u))$. In the horizontal case, we use Lemma 5.5 and the degree computation of Construction 4.2 to see that h^ζ and $\delta_{C,\beta}$ have the same K_0 -degree. Thus δ_κ is of K_0 -degree $Q_0(P^*(u))$. Since $P^*(u) \in \text{Ker}(Q)$ holds, we obtain that all δ_κ are of K -degree $Q(P^*(u)) = 0$. \square

Proposition 5.7. *Consider a minimally presented algebra $R(A, P)$ with its fine K_0 -grading and the coarser K -grading and let δ be a K_0 -homogeneous locally nilpotent derivation of K -degree zero on $R(A, P)$.*

- (i) *If δ is of vertical type, then there is an index $1 \leq k_0 \leq m$ such that δ is a linear combination of derivations δ_{κ_t} with Demazure P -roots $\kappa_t = (u_t, k_0)$.*
- (ii) *If δ is of horizontal type, then are indices $0 \leq i_0, i_1 \leq r$ and a sequence $C = (c_0, \dots, c_r)$ such that δ is a linear combination of derivations δ_{κ_t} with Demazure P -roots $\kappa_t = (u_t, i_0, i_1, C)$.*

Lemma 5.8. *Let δ be a nontrivial K_0 -homogeneous locally nilpotent derivation on a minimally presented algebra $R(A, P)$ and let $r \geq 2$. If δ is of K -degree zero, then δ is primitive with respect to the K_0 -grading.*

Proof. We have to show that the K_0 -degree w of δ does not lie in the weight cone of the K_0 -grading. First observe that $w \neq 0$ holds: otherwise Corollary 3.6 yields that δ annihilates all generators T_{ij} and S_k , a contradiction to $\delta \neq 0$. Now assume that w lies in the weight cone of the K_0 -grading. Then, for some $d > 0$, we find a nonzero $f \in R(A, P)_{dw}$. The K -degree of f equals zero and thus f is constant, a contradiction. \square

Proof of Proposition 5.7. First assume that δ is vertical. Lemma 5.8 tells us that δ is primitive with respect to the K_0 -grading. According to Theorem 4.3, there is an index $1 \leq k_0 \leq m$ and an element $h \in R(A, P)$ represented by a polynomial only depending on variables from $\ker(\delta)$ such that we have

$$\delta(T_{ij}) = 0 \text{ for all } i, j, \quad \delta(S_k) = 0 \text{ for all } k \neq k_0, \quad \delta(S_{k_0}) = hS_{k_0}.$$

Clearly, h is K_0 -homogeneous of K -degree zero. Lemma 5.5 shows that the monomials of h are of the form h^u with $u \in M$. The fact that the monomials $h^u S_{k_0}$ do not depend on S_{k_0} and have nonnegative exponents yield the inequalities of a vertical Demazure P -root for each (u, k_0) . Consequently, δ is a linear combination of derivations arising from vertical Demazure P -roots.

We turn to the case that δ is horizontal. Again by Lemma 5.8, our δ is primitive with respect to the K_0 -grading and by Theorem 4.3 it has the form $h\delta_{C,\beta}$ for some K_0 -homogeneous $h \in \ker(\delta_{C,\beta})$. By construction, $\delta_{C,\beta}$ is induced by a homogeneous derivation of $\mathbb{K}[T_{ij}, S_k]$ having the same K_0 - and K -degrees; we denote this lifted derivation again by $\delta_{C,\beta}$. Similarly, h is represented by a polynomial in $\mathbb{K}[T_{ij}, S_k]$ which we again denote by h .

We show that any monomial of h depends only on variables from $\ker(\delta_{C,\beta})$. Indeed, suppose that there occurs a monomial $T_{ij}h'$ with $\delta_{C,\beta}(T_{ij}) \neq 0$ in h . Then, using the fact that δ is of K -degree zero, we obtain

$$\deg(T_{ij}) = \deg(\delta(T_{ij})) = \deg(T_{ij}) + \deg(h') + \deg(\delta_{C,\beta}(T_{ij})).$$

This implies $\deg(h') + \deg(\delta_{C,\beta}(T_{ij})) = 0$; a contradiction to the fact that the weight cone of the K -grading contains no lines. This proves the claim. Thus, we may assume that the polynomial h is a monomial.

The next step is to see that it is sufficient to take derivations $\delta_{C,\beta}$ with a vector β in the row space having one zero coordinate. Consider a general β , that means one with only nonvanishing coordinates. By construction, the row space of A contains unique vectors β^0 and β^1 with $\beta_0^0 = \beta_1^1 = 0$ and $\beta = \beta^0 + \beta^1$. With these vectors, we have

$$h\delta_{C,\beta} = h\frac{\partial T_0^{l_0}}{\partial T_{0c_0}}\delta_{C,\beta^0} + h\frac{\partial T_1^{l_1}}{\partial T_{1c_1}}\delta_{C,\beta^1}.$$

By Construction 4.2, the K_0 -degrees and thus the K -degrees of the left hand side and of the summands coincide. Moreover, h is a monomial in generators from $\ker(\delta_{C,\beta})$ and any such generator is annihilated by δ_{C,β^0} and by δ_{C,β^1} too.

Let $e = (e_{ij}, e_k)$ denote the exponent vector of the monomial h . According to Lemma 5.5, the condition that the (K_0 -homogeneous) derivation δ has K -degree zero is equivalent to the fact that the monomial

$$T_{i_1c_{i_1}}^{-1} h\delta_{C,\beta}(T_{i_1c_{i_1}}) = T_{i_1c_{i_1}}^{-1} T_{i_0c_{i_0}}^{e_{i_0c_{i_0}}} \prod_{\substack{i \\ j \neq c_i}} T_{ij}^{e_{ij}} \prod_k S_k^{e_k} \beta_{i_1} \prod_{i \neq i_0, i_1} \frac{\partial T_i^{l_i}}{\partial T_{ic_i}}$$

has the form h^u for some linear form $u \in M$. Taking into account that the exponents e_{ij} and e_k are nonnegative, we see that these conditions are equivalent to equalities and inequalities in the definition of a horizontal Demazure P -root. \square

We recall the correspondence between locally nilpotent derivations and one parameter additive subgroups. Consider any integral affine \mathbb{K} -algebra R , where \mathbb{K} is an algebraically closed field of characteristic zero. Every locally nilpotent derivation $\delta: R \rightarrow R$ gives rise to a rational representation $\varrho_\delta: \mathbb{G}_a \rightarrow \text{Aut}(R)$ of the additive group \mathbb{G}_a of the field \mathbb{K} via

$$\varrho_\delta(t)(f) := \exp(t\delta)(f) := \sum_{d=0}^{\infty} \frac{t^d}{d!} \delta^d(f).$$

This sets up a bijection between the locally nilpotent derivations of R and the rational representations of \mathbb{G}_a by automorphisms of R . The representation associated to a locally nilpotent derivation $\delta: R \rightarrow R$ gives rise to a one parameter additive subgroup (1-PASG) of the automorphism group of $\overline{X} := \text{Spec } R$:

$$\lambda_\delta: \mathbb{G}_a \rightarrow \text{Aut}(\overline{X}), \quad t \mapsto \text{Spec}(\varrho_\delta(t)).$$

Now suppose that R is graded by some finitely generated abelian group K_0 and consider the associated action of $H_0 := \text{Spec } \mathbb{K}[K_0]$ on $\overline{X} = \text{Spec } R$. We relate homogeneity of locally nilpotent derivation δ to properties of the associated subgroup $\overline{U}_\delta := \lambda_\delta(\mathbb{G}_a)$ of $\text{Aut}(\overline{X})$.

Lemma 5.9. *In the above setting, let δ be a locally nilpotent derivation on R . The following statements are equivalent.*

- (i) *The derivation δ is K_0 -homogeneous.*
- (ii) *One has $h\overline{U}_\delta h^{-1} = \overline{U}_\delta$ for all $h \in H_0$.*

Moreover, if one of these two statements holds, then the degree $w := \deg(\delta) \in K_0$ is uniquely determined by the property

$$h\varrho_\delta(t)h^{-1} = \varrho_\delta(\chi^w(h)t) \text{ for all } h \in H_0.$$

Proof of Theorem 5.4. Assertion (i) is clear by Corollary 2.4 the fact that X is nontoric. We prove (ii). Consider $R(A, P)$ with its fine K_0 -grading and the coarser K -grading. The quasitori $H_0 := \text{Spec } \mathbb{K}[K_0]$ and $H := \text{Spec } \mathbb{K}[K]$ act effectively on $\overline{X} = \text{Spec } R(A, P)$. We view H_0 and H as subgroups of $\text{Aut}(\overline{X})$. For any locally nilpotent derivation δ on $R(A, P)$ and $\overline{U}_\delta = \lambda_\delta(\mathbb{G}_a)$, Lemma 5.9 gives

$$\begin{aligned} \delta \text{ is } K_0\text{-homogeneous} &\iff h\overline{U}_\delta h^{-1} = \overline{U}_\delta \text{ for all } h \in H_0, \\ \delta \text{ is } K\text{-homogeneous of degree } 0 &\iff huh^{-1} = u \text{ for all } h \in H, u \in \overline{U}_\delta. \end{aligned}$$

Recall that X arises as $X = \widehat{X} // H$ for an open H_0 -invariant set $\widehat{X} \subseteq \overline{X}$. Moreover, the action of $T = H_0/H$ on X is the induced one, i.e. it makes the quotient map $p: \widehat{X} \rightarrow X$ equivariant. Set for short

$$\overline{G} := \text{CAut}(\overline{X}, H)^0, \quad G := \text{Aut}(X)^0.$$

Denote by $1\text{-PASG}_{H_0}(\overline{G})$ and $1\text{-PASG}_T(G)$ the one parameter additive subgroups normalized by H_0 and T respectively. Moreover, let $\text{LND}(R(A, P))_0$ denote the K -homogeneous locally nilpotent derivations of K -degree zero and $\text{LND}_{K_0}(R(A, P))_0$ the subset of K_0 -homogeneous ones. Then we arrive at a commutative diagram

$$\begin{array}{ccc} \text{LND}_{K_0}(R(A, P))_0 & \subseteq & \text{LND}(R(A, P))_0 \\ \cong \updownarrow & & \updownarrow \cong \\ 1\text{-PASG}_{H_0}(\overline{G}) & \subseteq & 1\text{-PASG}(\overline{G}) \\ p_* \downarrow & & \downarrow p_* \\ 1\text{-PASG}_T(G) & \subseteq & 1\text{-PASG}(G) \end{array}$$

Construction 5.6 associates an element $\delta_\kappa \in \text{LND}_{K_0}(R(A, P))_0$ to any Demazure P -root κ . Going downwards the left hand side of the above diagram, the latter turns into an element $\lambda_\kappa \in 1\text{-PASG}_T(G)$. Differentiation gives the T -eigenvector $\dot{\lambda}_\kappa(1) \in \text{Lie}(G)$ having as its associated root the unique character χ of T satisfying

$$t\lambda_\kappa(z)t^{-1} = \lambda_\kappa(\chi(t)z) \quad \text{for all } t \in T, z \in \mathbb{K}.$$

Remark 3.9 and Lemma 5.9 show that under the identification $\mathbb{X}(T) = \mathbb{Z}^s$ the character χ is just the \mathbb{Z}^s -part of the Demazure P -root κ . Proposition 5.7 tells us that any element of $\text{LND}_{K_0}(R(A, P))_0$ is linear combination of derivations δ_κ arising from Demazure P -roots. Moreover, by Corollary 2.7, the push forward p_* maps $1\text{-PASG}_{H_0}(\overline{G})$ onto $1\text{-PASG}_T(G)$. We conclude that $\text{Lie}(G)$ is generated by $\text{Lie}(T)$ and $\dot{\lambda}_\kappa(1)$, where κ runs through the P -Demazure roots. Assertion (ii) follows. \square

Corollary 5.10 (of proof). *Let X be a nontoric normal complete rational variety with a torus action $T \times X \rightarrow X$ of complexity one arising according to Construction 3.10 from $R(A, P)$ by a good quotient $p: \widehat{X} \rightarrow X$. Then every Demazure P -root κ induces a one parameter subgroup $\lambda_\kappa = p_*\lambda_{\delta_\kappa}: \mathbb{G}_a \rightarrow \text{Aut}(X)$ and $\text{Aut}(X)^0$ is generated by T and the images $\lambda_\kappa(\mathbb{G}_a)$.*

Example 5.11 (The E_6 -singular cubic V). Let A and P as in Example 5.2. From there we infer that $R(A, P)$ admits precisely one horizontal Demazure P -root. For the automorphism group of the corresponding surface X this means that $\text{Aut}(X)^0$ is the semidirect product of \mathbb{K}^* and \mathbb{G}_a twisted via the weight 3, see again 5.2. In particular, the surface X is quasihomogeneous. Finally, in this case, one can show

directly that the group of graded automorphisms of $R(A, P)$ is connected. Thus, Theorem 2.1 yields that $\text{Aut}(X)$ is connected.

6. ALMOST HOMOGENEOUS SURFACES

A variety is *almost homogeneous* if its automorphism group acts with an open orbit. We take a closer look to this case with a special emphasis on almost homogeneous rational \mathbb{K}^* -surfaces of Picard number one. First let us see how they fit into the class of almost homogeneous Mori dream surfaces.

Proposition 6.1. *Let X be an almost homogeneous Mori dream surface. Then one of the following statements holds:*

- (i) X is toric, i.e. an equivariant compactification of $\mathbb{K}^* \times \mathbb{K}^*$,
- (ii) $\text{Aut}(X)$ has \mathbb{K}^* as its maximal torus,
- (iii) $\text{Aut}(X)^0$ is unipotent.

Proof. According to Theorem 2.1, the automorphism group $\text{Aut}(X)$ is a linear algebraic group. Let $T \subseteq \text{Aut}(X)$ be a maximal torus. Then the cases (i), (ii) and (iii) correspond to the cases $\dim(T) = 2, 1, 0$. \square

The following statement characterizes the almost homogeneous varieties coming with a torus action of complexity one in arbitrary dimension.

Theorem 6.2. *Let X be a nontoric normal complete rational variety with a torus action $T \times X \rightarrow X$ of complexity one and Cox ring $\mathcal{R}(X) = R(A, P)$. Then the following statements are equivalent.*

- (i) *The variety X is almost homogeneous.*
- (ii) *There exists a horizontal Demazure P -root.*

Moreover, if one of these statements holds, and $R(A, P)$ is minimally presented, then the number $r - 1$ of relations of $R(A, P)$ is bounded by

$$r - 1 \leq \dim(X) + \text{rk}(\text{Cl}(X)) - m - 2.$$

Proof. If (i) holds, then $\text{Aut}(X)$ acts with an open orbit on X . According to Corollary 5.10, there must be a Demazure P -root κ such that the semidirect product $T \ltimes U$ acts with an open orbit on X , where $U = p_*(\delta_\kappa(\mathbb{G}_a))$ is one-dimensional unipotent subgroup of $\text{Aut}(X)$ associated to κ . Since U acts nontrivially on the field $\mathbb{K}(X)^T$ of T -invariant functions, the Demazure P -root must be horizontal.

Conversely, if (ii) holds, then take a horizontal Demazure P -root κ . The associated one-dimensional unipotent subgroup $U = p_*(\delta_\kappa(\mathbb{G}_a))$ of $\text{Aut}(X)$ acts nontrivially on the field $\mathbb{K}(X)^T$ of T -invariant functions and thus $T \ltimes U$ acts with an open orbit on X .

For the supplement, recall first that $R(A, P)$ is a complete intersection with $r - 1$ necessary relations and thus we have

$$n + m - (r - 1) = \dim(R(A, P)) = \dim(X) + \text{rk}(\text{Cl}(X)).$$

Now observe that any relation g_I involving only three variables prevents existence of a horizontal Demazure P -root. Consequently, by suitably arranging the relations, we have $n_0, n_1 \geq 1$ and $n_i \geq 2$ for all $i \geq 2$. Thus, $n \geq 2 + 2(r - 1)$ holds and the assertion follows. \square

We specialize to dimension two. Any normal complete rational \mathbb{K}^* -surface X is determined by its Cox ring and thus is given up to isomorphism by the defining data A and P of the ring $\mathcal{R}(X) = R(A, P)$; we also say that the \mathbb{K}^* -surface X arises from A and P and refer to [8, Sec. 3.3] for more background. A first step towards the almost homogeneous X is to determine possible horizontal Demazure P -roots in the following setting.

Proposition 6.3. *Consider integers $l_{02} \geq 1$, $l_{11} \geq l_{21} \geq 2$ and d_{01} , d_{02} , d_{11} , d_{21} such that the following matrix has pairwise different primitive columns generating \mathbb{Q}^3 as a convex cone:*

$$P := \begin{bmatrix} -1 & -l_{02} & l_{11} & 0 \\ -1 & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}.$$

Moreover, assume that P is positive in the sense that $\det(P_{01}) > 0$ holds, where P_{01} is the 3×3 matrix obtained from P by deleting the first column. Then the possible horizontal Demazure P -roots are

- (i) $\kappa = (u, 1, 2, (1, 1, 1))$, where $u = \left(d_{01}\mu + \frac{d_{21}\mu+1}{l_{21}}, -\frac{d_{21}\mu+1}{l_{21}}, \mu\right)$ with an integer μ satisfying

$$l_{21} \mid d_{21}\mu + 1, \quad \frac{l_{02}}{d_{02} - l_{02}d_{01}} \leq \mu \leq -\frac{l_{11}}{l_{21}d_{11} + l_{11}d_{21} + d_{01}l_{11}l_{21}},$$

- (ii) if $l_{02} = 1$: $\kappa = (u, 1, 2, (2, 1, 1))$, where $u = \left(d_{02}\mu + \frac{d_{21}\mu+1}{l_{21}}, -\frac{d_{21}\mu+1}{l_{21}}, \mu\right)$ with an integer μ satisfying

$$l_{21} \mid d_{21}\mu + 1, \quad -\frac{l_{11}}{l_{21}d_{11} + l_{11}d_{21} + d_{02}l_{11}l_{21}} \leq \mu \leq \frac{1}{d_{01} - d_{02}},$$

- (iii) $\kappa = (u, 2, 1, (1, 1, 1))$, where $u = \left(-\frac{d_{11}\mu+1}{l_{11}}, d_{01}\mu + \frac{d_{11}\mu+1}{l_{11}}, \mu\right)$ with an integer μ satisfying

$$l_{11} \mid d_{11}\mu + 1, \quad \frac{l_{02}}{d_{02} - l_{02}d_{01}} \leq \mu \leq -\frac{l_{21}}{l_{21}d_{11} + l_{11}d_{21} + d_{01}l_{11}l_{21}},$$

- (iv) if $l_{02} = 1$: $\kappa = (u, 2, 1, (2, 1, 1))$, where $u = \left(-\frac{d_{11}\mu+1}{l_{11}}, d_{02}\mu + \frac{d_{11}\mu+1}{l_{11}}, \mu\right)$ with an integer μ satisfying

$$l_{11} \mid d_{11}\mu + 1, \quad -\frac{l_{21}}{l_{21}d_{11} + l_{11}d_{21} + d_{02}l_{11}l_{21}} \leq \mu \leq \frac{1}{d_{01} - d_{02}}.$$

Proof. In the situation of (i), evaluating the general linear form $u = (u_1, u_2, u_3)$ on the columns of P gives the following conditions for a Demazure P -root:

$$-u_1 - u_2 + u_3d_{01} = 0, \quad u_2l_{21} + u_3d_{21} = -1,$$

$$-u_1l_{02} - u_2l_{02} + u_3d_{02} \geq l_{02}, \quad u_1l_{11} + u_3d_{11} \geq 0.$$

Resolving the equations for u_1, u_2 and plugging the result into the inequalities gives the desired roots with $\mu := u_3$. The other cases are treated analogously. \square

Corollary 6.4. *The nontoric almost homogeneous normal complete rational \mathbb{K}^* -surfaces X of Picard number one are precisely the ones arising from data*

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -l_{02} & l_{11} & 0 \\ -1 & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}$$

as in 6.3 allowing an integer μ according to one of the Conditions 6.3 (i) to (iv). In particular, the Cox ring of X is given as

$$\mathcal{R}(X) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] / \langle T_{01}T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}} \rangle$$

with the grading by $\mathbb{Z}^4/\text{im}(P^*)$. Moreover, the anticanonical divisor of X is ample, i.e. X is a del Pezzo surface.

Proof. As any surface with finitely generated Cox ring, X is \mathbb{Q} -factorial. Since X has Picard number one, the divisor class group $\text{Cl}(X)$ is of rank one. Now take a minimal presentation $\mathcal{R}(X) = R(A, P)$ of the Cox ring. Then, according to Theorem 6.2, we have exactly one relation in $R(A, P)$ and thus P is a 3×4 matrix. Moreover, Theorem 6.2 says that there is a horizontal Demazure P -root. Consequently, one of the exponents l_{01} and l_{02} must equal one, say l_{01} . Fixing a suitable order for the last two variables we ensure $l_{11} \geq l_{21}$. Passing to the \mathbb{K}^* -action $t^{-1} \cdot x$ instead of $t \cdot x$ if necessary, we achieve that P is positive in the sense of 6.3.

Let us see why X is a del Pezzo surface. Denote by P_{ij} the matrix obtained from P by deleting the column v_{ij} . Then, in $\text{Cl}(X)^0 = \mathbb{Z}$, the factor group of $\text{Cl}(X)$ by the torsion part, the weights w_{ij}^0 of T_{ij} are given up to a factor α as

$$(w_{01}^0, w_{02}^0, w_{11}^0, w_{21}^0) = \alpha(\det(P_{01}), -\det(P_{02}), \det(P_{11}), -\det(P_{21})).$$

According to [1, Prop. III.3.4.1], the class of the anticanonical divisor in $\text{Cl}(X)^0$ is given as the sum over all w_{ij}^0 , minus the degree of the relation. The inequalities on the l_{ij}, d_{ij} implied by the existence of an integer μ as in 6.3 (i) to (iv) show that the anticanonical class is positive (note that α rules out). \square

We turn to the case of precisely one singular point. The previously diophantic aspect in the condition on existence of Demazure P -roots then disappears: no divisibility condition remains.

Construction 6.5 (\mathbb{K}^* -surfaces with one singularity). Consider a triple (l_0, l_1, l_2) of integers satisfying the following conditions:

$$l_0 \geq 1, \quad l_1 \geq l_2 \geq 2, \quad l_0 \leq l_1 l_2, \quad \gcd(l_1, l_2) = 1.$$

Let (d_1, d_2) be the (unique) pair of integers with $d_1 l_2 + d_2 l_1 = -1$ and $0 \leq d_2 < l_2$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -l_0 & l_1 & 0 \\ -1 & -l_0 & 0 & l_2 \\ 0 & 1 & d_1 & d_2 \end{bmatrix}$$

Then the associated ring $R(l_0, l_1, l_2) := R(A, P)$ is graded by $\mathbb{Z}^4 / \text{im}(P^*) \cong \mathbb{Z}$, and is explicitly given by

$$R(l_0, l_1, l_2) = \mathbb{K}[T_1, T_2, T_3, T_4] / \langle T_1 T_2^{l_0} + T_3^{l_1} + T_4^{l_2} \rangle, \\ \deg(T_1) = l_1 l_2 - l_0, \quad \deg(T_2) = 1, \quad \deg(T_3) = l_2, \quad \deg(T_4) = l_1.$$

Proposition 6.6. *For the \mathbb{K}^* -surface $X = X(l_0, l_1, l_2)$ with Cox ring $R(l_0, l_1, l_2)$, the following statements hold:*

- (i) X is nontoric and we have $\text{Cl}(X) = \mathbb{Z}$,
- (ii) X comes with precisely one singularity,
- (iii) X is a del Pezzo surface if and only if $l_0 < l_1 + l_2 + 1$ holds,
- (iv) X is almost homogeneous if and only if $l_0 \leq l_1$ holds.

Moreover, any normal complete rational nontoric \mathbb{K}^* -surface of Picard number one with precisely one singularity is isomorphic to some $X(l_0, l_1, l_2)$.

Proof. First note that $X = X(l_0, l_1, l_2)$ is obtained as in Construction 3.10: the group $H_X = \mathbb{K}^*$ acts on \mathbb{K}^4 by

$$t \cdot z = (t^{l_1 l_2 - l_0} z_1, t z_2, t^{l_2} z_3, t^{l_1} z_4),$$

the total coordinate space $\overline{X} := V(T_1 T_2^{l_0} + T_3^{l_1} + T_4^{l_2})$ is invariant under this action and we have

$$\widehat{X} = \overline{X} \setminus \{0\}, \quad X = \widehat{X} / \mathbb{K}^*.$$

Thus, $\text{Cl}(X) = \mathbb{Z}$ holds and, since the Cox ring $\mathcal{R}(X) = R(l_0, l_1, l_2)$ is not a polynomial ring, X is nontoric.

Using [1, Prop. III.3.1.5], we now determine the singularities of X . The local divisor class group of the image $[1, 0, 0, 0] \in X$ of the point $(1, 0, 0, 0) \in \widehat{X}$ under the quotient map $\widehat{X} \rightarrow X$ is given by

$$\mathrm{Cl}(X, [1, 0, 0, 0]) = \mathbb{Z}/(l_1 l_2 - l_0)\mathbb{Z}.$$

This group is nontrivial and thus $[1, 0, 0, 0]$ is singular. All other local divisor class groups of X are trivial. Since moreover, all the singular points of \widehat{X} lie in the orbit $\mathbb{K}^* \cdot (1, 0, 0, 0)$, we conclude that $[1, 0, 0, 0] \in X$ is the only singular point.

According to [1, Prop. III.3.4.1], the anticanonical class of X is $l_1 + l_2 + 1 - l_0$. This proves (iii). Finally, for (iv), we infer from Proposition 6.3 that $\mu = l_1$ gives the desired Demazure P -root.

We come to the supplement. The surface X arises from a ring $R(A, P)$, where we may assume that $R(A, P)$ is minimally presented. The first task is to show that $n = 4$, $m = 0$ and $r = 2$ holds. We have

$$n + m - (r - 1) = \dim(X) + \mathrm{rk}(\mathrm{Cl}(X)) = 3.$$

Any relation g_I involving only three variables gives rise to a singularity in the source and a singularity in the sink of the \mathbb{K}^* -action. We conclude that at most two of the monomials occurring in the relations may depend only on one variable. Thus, the above equation shows $n = 4$, $m = 0$ and $r = 2$.

We may assume that the defining equation is of the form $T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}$. Again, since one of the two elliptic fixed points must be smooth, we can conclude that one of l_{0j} equals one, say l_{01} . Now it is a direct consequence of the description of the local divisor class groups given in [1, Prop. III.3.1.5] that a \mathbb{K}^* -surface with precisely one singularity arises from a matrix P as in the assertion. \square

Now we look at the log terminal ones of the $X(l_0, l_1, l_2)$; recall that a singularity is *log terminal* if all its resolutions have discrepancies bigger than -1 . Over \mathbb{C} , the log terminal surface singularities are precisely the quotient singularities by subgroups of $\mathrm{Gl}_2(\mathbb{C})$. The *Gorenstein index* of X is the minimal positive integer $\iota(X)$ such that $\iota(X)$ times the canonical divisor \mathcal{K}_X is Cartier.

Corollary 6.7. *Assume that $X = X(l_1, l_2, l_3)$ is log terminal. Then we have the following three cases:*

- (i) *the surface X is almost homogeneous,*
- (ii) *the singularity of X is of type E_7 ,*
- (iii) *the singularity of X is of type E_8 .*

Moreover, for the almost homogeneous surfaces $X = X(l_1, l_2, l_3)$ of Gorenstein index $\iota(X) = a$, we have

- (i) $(l_0, l_1, l_2) = (1, l_1, l_2)$ with the bounds $l_2 \leq l_1 \leq 2a^2 + \frac{4}{3}a$,
- (ii) $(l_0, l_1, l_2) = (2, l_1, 2)$ with the bound $l_1 \leq 3a + 2$,
- (iii) $(l_0, l_1, l_2) = (3, 3, 2), (2, 4, 3), (2, 5, 3), (3, 5, 2)$.

Proof. The condition that X is log terminal means that the number $l_0 l_1 l_2$ is bounded by $l_0 l_1 + l_0 l_2 + l_1 l_2$; this can be seen by explicitly performing the canonical resolution of singularities of $X(l_0, l_1, l_2)$, see [8, Sec. 3]. Thus, the allowed (l_0, l_1, l_2) must be platonic triples and we are left with

$$(1, l_1, l_2), (2, l_1, 2), (3, 3, 2), (2, 4, 3), (2, 5, 3), (3, 5, 2), (4, 3, 2), (5, 3, 2).$$

The last two give the surfaces with singularities E_7 , E_8 and in all other cases, the resulting surface is almost homogeneous by Proposition 6.6. The Gorenstein condition says that $a\mathcal{K}_X$ lies in the Picard group. According to [1, Cor. III.3.1.6], this is equivalent to the fact that $l_1 l_2 - l_0$ divides $a \cdot (l_1 + l_2 + 1 - l_0)$. The bounds then follow by elementary estimations. \square

Corollary 6.8. *The following tables lists the triples (l_0, l_1, l_2) together with roots of $\text{Aut}(X)$ for the log terminal almost homogeneous complete rational \mathbb{K}^* -surfaces $X = X(l_0, l_1, l_2)$ with precisely one singularity up to Gorenstein index $\iota(X) = 5$.*

$\iota(X) = 1$	$\iota(X) = 2$	$\iota(X) = 3$
$(1, 3, 2) : \{1, 2, 3\}$	$(1, 7, 3) : \{1, 3, 4, 7\}$	$(2, 7, 2) : \{2, 3, 5, 7\}$
$(2, 3, 2) : \{2, 3\}$		$(1, 13, 4) : \{1, 4, 5, 9, 13\}$
$(3, 3, 2) : \{3\}$		$(1, 8, 5) : \{3, 5, 8\}$
$\iota(X) = 4$	$\iota(X) = 5$	
$(2, 5, 2) : \{2, 3, 5\}$	$(2, 11, 2) : \{2, 3, 5, 7, 9, 11\}$	
$(1, 21, 5) : \{1, 5, 6, 11, 16, 21\}$	$(1, 13, 7) : \{2, 6, 13\}$	
	$(2, 4, 3) : \{3, 4\}$	
	$(1, 17, 3) : \{2, 3, 5, 8, 11, 14, 17\}$	
	$(1, 31, 6) : \{1, 6, 7, 13, 19, 25, 31\}$	
	$(1, 18, 7) : \{4, 7, 11, 18\}$	

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